

Wednesday, November 5, Lecture 28: Fourier Series and the Spectral Theorem

Last time...

- $\{\delta_k : k \in \mathbb{Z}\}$ is an orthonormal basis of $\ell^2(\mathbb{Z})$,
- $\{e^{ikx} : k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2([-\pi, \pi])$,

so we may write any $f \in L^2([-\pi, \pi])$ as

$$\begin{aligned} f(x) &= \sum_{k=-\infty}^{\infty} \langle e^{ikx}, f \rangle e^{ikx} \\ &= \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx} \end{aligned}$$

- Really, this " $=$ " means they are the same as L^2 functions,
$$\lim_{N \rightarrow \infty} \|f - \sum_{k=-N}^N \hat{f}(k) e^{ikx}\| = 0$$

Could happen that at specific x , this series diverges.*

- Parseval's Identity: $U: L^2([-\pi, \pi]) \rightarrow \ell^2(\mathbb{Z})$ is
$$f \mapsto \hat{f} = Uf.$$

a unitary map, so

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} g(x) dx = \langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle = \sum_{k=-\infty}^{\infty} \overline{\hat{f}(k)} \hat{g}(k)$$

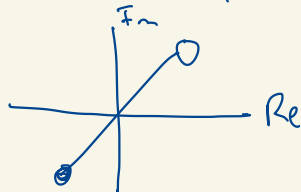
Special case:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \|f\|^2 = \|\hat{f}\|^2 = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2.$$

It may be that Fourier series does not converge pointwise, yet still converges in L^2 .

Ex: Sawtooth wave

$$f(x) = x, \quad -\pi \leq x < \pi$$



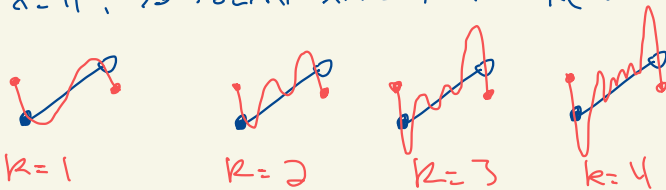
can find Fourier series to be

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{k} e^{ikx}$$

But this diverges at $x = \pi$, since $e^{i\pi k} = (-1)^k$, so

$$= \sum_{k=-\infty}^{\infty} \frac{1}{k} \quad \text{(It converges everywhere else)}$$

This is because sawtooth has a discontinuity at $x = \pi$, so Fourier series looks like (I'm drawing real part)



Gibbs phenomenon: Fourier series oscillates wildly near discontinuity and diverges at discontinuity, but still converges in L^2 (since "problem area" is $\emptyset$, since it's just at the point $x = \pi$)

$$\int_{-\pi}^{\pi} f(x) dx = 0.$$

Recall: Spectral Theorem

Let $A = A^\dagger$ (A is called Hermitian, or self-adjoint)

Then A has an orthonormal basis of eigenvectors,
i.e. there is a unitary change-of-basis matrix
which diagonalizes A :

$$\begin{aligned} A &= U D U^{-1} \\ &= U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^{-1} \end{aligned}$$

This still works on ∞ -dim Hilbert spaces (once we understand "diagonal" better.)

Now: let's think about some operators on $L^2([-\pi, \pi])$.

Ex: multiplication operators. $\leftarrow$ these will be "diagonal" operators!

Let $V \in L^2([-\pi, \pi])$. Then consider the linear operator

$M_V: L^2([-\pi, \pi]) \rightarrow L^2([-\pi, \pi])$ given by

$$(M_V f)(x) := V(x) f(x), \quad \forall x \in [-\pi, \pi].$$

• Physics note: if $V(x) = x$, this is the position operator $\hat{x}$.

Check:

• Linear? Let $a, b \in \mathbb{C}$ and $f, g \in L^2$.

$$\begin{aligned} (M_V (af + bg))(x) &= V(x) (af(x) + bg(x)) \\ &= a V(x) f(x) + b V(x) g(x) \\ &= a (M_V f)(x) + b (M_V g)(x) \end{aligned}$$

• Does it map $L^2([-\pi, \pi]) \rightarrow L^2([-\pi, \pi])$?

Yes:

$$\|M_V f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |V(x) f(x)|^2 dx$$

$$= \langle |V|, |f|^2 \rangle$$

Cauchy-Schwarz $\rightarrow \leq \|V\| \|f\|^2$

$< \infty$ since $V \in L^2, f \in L^2$.

Multiplication operators are like diagonal matrices,
If $v: \{1, 2, \dots, n\} \rightarrow \mathbb{C}$, then

$$M_v = \begin{bmatrix} v(1) & & \\ & \ddots & \\ & & v(n) \end{bmatrix}, \quad f = \begin{bmatrix} f(1) \\ \vdots \\ f(x) \\ \vdots \\ f(n) \end{bmatrix}$$

then $(M_v f)(x) = v(x) f(x), \quad x = 1, \dots, n$

• We love diagonal matrices

• Multiplication operators are self-adjoint if $V(x) = \overline{V(x)}$:

$$\begin{aligned} \langle f, M_v g \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} V(x) g(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{V(x) f(x)} g(x) dx \\ &= \langle M_v f, g \rangle. \end{aligned}$$

$$\Rightarrow M_v^* = M_v.$$

Ex: Derivative operators.

Consider $p : L^2([-\pi, \pi]) \rightarrow L^2([-\pi, \pi])$, which acts by

$$(pf)(x) = -i \frac{df}{dx}(x).$$

Technically, p only makes sense on differentiable $f \in L^2([-\pi, \pi])$. Mathematicians say p is "densely defined".

Demand: $f(-\pi) = f(\pi)$, periodic boundary conditions.

Then observe that p is also self-adjoint:

$$\langle f, pg \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} \left(-i \frac{dg}{dx}\right)(x) dx$$

Integration by parts $\rightarrow$

$$= \frac{1}{2\pi} \left(-i \left(\overline{f(\pi)} g(\pi) - \overline{f(-\pi)} g(-\pi) \right) + i \int_{-\pi}^{\pi} \overline{\frac{df}{dx}(x)} g(x) dx \right)$$

p.b.c. $\rightarrow$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{-i \frac{df}{dx}(x)} g(x) dx$$

$$= \langle pf, g \rangle.$$

So $p = p^\dagger$. Spectral theorem suggests we can diagonalize p , i.e. conjugate by unitary to get a multiplication operator.

Next time: The Fourier Transform $f(x) \xrightarrow{F} \hat{f}(k)$ is the unitary which diagonalizes p !

(Unitarity is Parseval's Theorem $\langle f, g \rangle = \langle F(f), F(g) \rangle = \langle \hat{f}, \hat{g} \rangle$)